On the irreducible module of quantum group $U_q(B_2)$ at a root of 1

1994 J. Phys. A: Math. Gen. 27 2401

(http://iopscience.iop.org/0305-4470/27/7/019)

J. Phys. A: Math. Gen. 27 (1994) 2401–2405. Printed in the UK

On the irreducible module of quantum group $U_q(B_2)$ at a root of 1

Shanyou Zhou
Department of Mathematics, Graduate School, Academia Sinica, PO Box 3908, Beijing 100039, People's Republic of China

Received 23 April 1993, in final form 21 December 1993

Abstract. This paper deals with the irreducible highest-weight module $L(\lambda)$ of quantum group $U_q(B_2)$ when q is a root of unity. The character of $L(\lambda)$ has been obtained in one of the cases. As a consequence, its dimension has also been obtained. In addition, a centre element of $U_q(B_2)$ has been found in explicit form.

1. Introduction

As the quotient module of the Verma module $V(\lambda)$ of $U_q(g)$, the irreducible highest-weight module $L(\lambda)$ is always one of the most interesting subjects in representation theory. In this paper, we will discuss the Verma module $V(\lambda)$ of $U_q(B_2)$ in section 1. The explicit expression of singular vectors under the canonical basis contained in $V(\lambda)$ has been given when $q^N = 1$. The embedding relations of the Verma proper submodule has been partly discussed. In section 2, the character of irreducible highest-weight module $L(\lambda)$ has been determined. As a consequence, its dimension has also been obtained.

2. Verma module of $U_q(B_2)$

The quantum group $U_q(B_2)$ with Cartan matrix

$$A = (a_{ij}) = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

is an associative algebra over the fraction field $\mathbb{C}(q)$, where q is an indeterminate. Its generators are $E_i, F_i, K_i, K_i^{-1}, i = 1, 2$ and the defined relations are

$$\begin{cases} K_i K_i^{-1} = K_i^{-1} K_i = 1 \\ K_i K_j = K_j K_i \\ K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j \\ K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j \\ E_i F_j - F_j E_i = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1}) \\ \displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^s E_j E_i^{1-a_{ij}-s} = 0 \qquad i \neq j \\ \displaystyle\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^s F_j F_i^{1-a_{ij}-s} = 0 \qquad i \neq j \end{cases} \tag{1}$$

for $i, j = 1, 2$, where $d_1 = 1, d_2 = 2, q_i = q^{d_i}$; the Gaussian binomial coefficients are

$$\begin{bmatrix} n \\ m \end{bmatrix}_{d_i} = [n]_{d_i}!/[m]_{d_i}![n-m]_{d_i}! \qquad \text{for } n, m \in \mathbb{N}$$

defined by $[n]_{d_i} = (q_i^n - q_i^{-n})/(q_i - q_i^{-1})$ and $[n]_{d_i}! = [n]_{d_i}[n-1]_{d_i} \cdots [2]_{d_i}[1]_{d_i}$. In particular, put $[n]_{d_i} = [n]$ if $d_i = 1$.

Lusztig [2] gives the canonical basis over $\mathbb{C}(q)$:

$$\{F_1^{k_1} F_{112}^{k_2} F_{12}^{k_3} F_2^{k_4} K_1^{l_1} K_2^{l_2} E_2^{r_4} E_{12}^{r_3} E_{112}^{r_2} E_1^{r_1} | k_i, r_i \in \mathbb{Z}_{\geqslant 0}, l_j \in \mathbb{Z}, 1 \leqslant i \leqslant 4, j = 1, 2\}$$

where

$$E_{12} = q^{-2}E_1E_2 - E_2E_1 \qquad E_{112} = [2]^{-1}(E_1E_{12} - E_{12}E_1)$$
$$F_{12} = q^2F_2F_1 - F_1F_2 \qquad F_{112} = [2]^{-1}(F_{12}F_1 - F_1F_{12}).$$

De Concini and Kac [1] find the method to compute the centre elements of quantum group $U_q(g)$ for complex simple Lie algebra g. Using their method, we can get all of the centre elements of $U_q(g)$. For instance, taking the initial term $\varphi_{00} = \{q^2K_2\}_q + \{q^4K_1^2K_2\}_q$, we get the explicit expression of the quadratic Casimir element C of $U_q(B_2)$ (see [1], section 2)

$$\begin{aligned} C = \{q^2K_2\}_q &+ \{q^4K_1^2K_2\}_q + (q - q^{-1})^2\{q^3K_1K_2\}_q F_1E_1 + [2]^2F_2E_2 \\ &- q^{-1}(q^2 - q^{-2})K_1^{-1}F_1F_2E_{12} + q(q^2 - q^{-2})K_1F_{12}E_2E_1 \\ &+ (q - q^{-1})^2\{q^3K_1\}_q F_{12}E_{12} - (q - q^{-1})^2F_1F_{12}E_{12}E_1 \\ &- q(q^2 - q^{-2})F_1F_{12}E_{112} + q^{-1}(q^2 - q^{-2})F_{112}E_{12}E_1 + [2]^2F_{112}E_{112} \end{aligned} \qquad (2)$$

where

$$\{x\}_q = \frac{(x + x^{-1})}{(q - q^{-1})^2}.$$

Using a fully different method, Zhang *et al* [4] have also obtained the same expression with a different constant term.

The Verma module $V(\lambda)$ with the highest-weight λ of quantum group $U_q(B_2)$ is generated by the so-called maximal vector v_0 such that $E_i \cdot v_0 = 0$, $K_i \cdot v_0 = q^{(\lambda|\alpha_i)}v_0$, $i = 1, 2$. From the canonical basis of $U_q(B_2)$, we can get the basis of $V(\lambda)$ easily

$$\{F_1^{k_1} F_{112}^{k_2} F_{12}^{k_3} F_2^{k_4} v_0 | k_i \in \mathbb{Z}_{\geqslant 0}, 1 \leqslant i \leqslant 4\}.$$

The vector $v_s \in V(\lambda)$ is called the singular vector if v_s is not its maximal vector v_0 and $E_i \cdot v_s = 0, i = 1, 2$. Obviously, if $v_s \in V(\lambda)$ is a singular vector, then it can generate a proper Verma submodule of $V(\lambda)$. Thus we have

Theorem 1.1. The Verma module $V(\lambda)$ is irreducible if and only if it does not contain any singular vector v_s.

For generic q, De Concini and Kac [1] pointed out that Verma module $V(\lambda)$ of $U_q(g)$ is irreducible if and only if $2(\lambda + \rho|\beta) \neq (\beta|\beta)$ for all $m \in \mathbb{N}$ and positive root β, where ρ is half of the sum of all positive roots of Lie algebra g.

But if q is the Nth primitive root of unity (for simplicity, let N be odd), then $F_1^N, F_{112}^N, F_{12}^N, F_2^N$ belong to the centre subalgebra of $U_q(B_2)$. Thus $F_1^{k_1N} F_{112}^{k_2N} F_{12}^{k_3N} F_2^{k_4N} v_0 \in V(\lambda)$ must be the singular vector, where $k_i \in \mathbb{Z}_{\geqslant 0}$ are not all zero. We will call them the singular vectors of type 1. Therefore if q is a root of 1, the Verma module $V(\lambda)$ is always reducible.

Theorem 1.2. Let $V(\lambda)$ be the Verma module with the highest weight λ. If q is the Nth root of 1 and N is odd, then the congruence equations associated with the Verma module $V(\lambda)$

$$2(\lambda+\rho|\beta_i) \equiv r_i(\beta_i|\beta_i)(\mathrm{mod}\, N) \qquad i=1,2,3,4 \tag{3}$$

(where $\beta_1=\alpha_1, \beta_2=2\alpha_1+\alpha_2, \beta_3=\alpha_1+\alpha_2, \beta_4=\alpha_2, \alpha_1$ is the short root, α_2 is the long root of B_2) have non-zero solution r_i if and only if the Verma module $V(\lambda)$ contains one-dimensional singular vector $v_s^{(i)}$ which is not of type 1.

Proof. Put $v_s^{(i)}$ in the form of the canonical basis. Solve the equations $E_j v_s^{(i)}=0$, $j=1,2$. We can get the explicit expression of one-dimensional singular vector $v_s^{(i)}$ as follows:

$$v_s^{(1)}=F_1^{r_1}v_0 \qquad v_s^{(2)}=\varphi_2^{(0)}(r_2)v_0 \qquad v_s^{(3)}=\varphi_3^{(0)}(r_3)v_0 \qquad v_s^{(4)}=F_2^{r_4}v_0$$

where

$$\varphi_2^{(0)}(r_2)=\sum_{s=0}^{2r_2}\sum_{0\leqslant 2t\leqslant s, s\leqslant r_2+t} a_{(s,t)}F_1^{2r_2-s}F_{112}^{t}F_{12}^{s-2t}F_2^{r_2-s+t}$$

the coefficients $a_{(s,t)}\in\mathbb{C}(q)$ satisfy the relations

$$a_{(s+1,t)}=a_{(s,t)}q^{(\lambda|\alpha_1)+2(r_2-s+t-1)}[2(r_2-s+t)][(\lambda|\alpha_1)+1-s]/[2][s-2t+1] \tag{4}$$

$$a_{(s,t+1)}=a_{(s,t)}q^{-2(r_2-s+2t)}[2]^2[(s-2t)][s-2t-1]/[2(r_2-s+t+1)][2(t+1)] \tag{5}$$

and $\varphi_3^{(0)}(r_3)=\sum_{s=0}^{r_3}\sum_{0\leqslant 2t\leqslant s}a_{(s,t)}F_1^{r_3-s}F_{112}^{t}F_{12}^{s-2t}F_2^{r_3-s+t}$, the coefficients $a_{(s,t)}\in\mathbb{C}(q)$ satisfy the relations

$$a_{(s+1,t)}=a_{(s,t)}q^{(\lambda|\alpha_1)+2(r_3-s+t-1)}[r_3-s][(\lambda|\alpha_1)+r_3-2s+2t+1]/[2][s-2t+1] \tag{6}$$

$$a_{(s,t+1)}=a_{(s,t)}q^{-(\lambda|\alpha_1)-(r_3-2s+4t+1)}[2]^2[s-2t][s-2t-1]/[(\lambda|\alpha_1)+r_3-2s+2t+3][2(t+1)]. \tag{7}$$

Remark. Dobrev [3] has got the explicit formula for the singular vectors of $V(\lambda)$ of quantum group $U_q(g)$ for complex simple Lie algebra g under another basis.

Replacing v_0 by $v_s^{(i)}$, we can find the new generation $v_s^{(ij)}$ of singular vectors, etc. If the congruence equation (3) has zero solution, then the corresponding singular vector is of type 1.

Denote by $V^{(i)}$ the Verma submodule generated by $v_s^{(i)}$ and by $V^{(ij)}$ the Verma submodule generated by $v_s^{(ij)}$ and so forth.

It is clear that we have the partly embedding relation of proper Verma submodules according to the relations of singular vectors.

Theorem 1.3. If the solutions r_i $(1\leqslant i\leqslant 4)$ of the congruence equations (3) satisfy $0<r_i<N, r_2=r_1+r_4$ and $r_3=r_1+2r_4$, then

(i) $V(\lambda)\supset V^{(1)}+V^{(4)}$
(ii) $V^{(1)}\cap V^{(4)}\supset V^{(12)}+V^{(43)}$
(iii) $V^{(12)}\cap V^{(43)}\supset V^{(2)}+V^{(3)}$
(iv) $V^{(2)}\cap V^{(3)}\supset V^{(1234)}$.

Proof. It is clear for (i). Note that there are the relations of the singular vectors $v_s^{(12)}=v_s^{(41)}\in V^{(1)}\cap V^{(4)}$ and $v_s^{(43)}=v_s^{(14)}\in V^{(1)}\cap V^{(4)}$; $v_s^{(2)}=v_s^{(123)}=v_s^{(431)}\in V^{(12)}\cap V^{(43)}$ and $v_s^{(3)}=v_s^{(432)}=v_s^{(124)}\in V^{(12)}\cap V^{(43)}$; $v_s^{(1234)}=v_s^{(24)}=v_s^{(31)}\in V^{(2)}\cap V^{(3)}$. The arguments (ii), (iii) and (iv) hold.

3. Irreducible module $L(\lambda)$

If the Verma module $V(\lambda)$ is reducible and J is its maximal proper submodule, then the quotient space $L(\lambda) = V(\lambda)/J$ as a $U_q(B_2)$-module is an irreducible highest-weight module.

Theorem 2.1. In the condition of theorem 1.3, the maximal proper submodule of $V(\lambda)$ of $U_q(B_2)$ is $V^{(1)} + V^{(4)}$.

Proof. We only need to prove that all of the singular vectors of type 1 are contained in the maximal proper submodule $V^{(1)} + V^{(4)}$ of $V(\lambda)$. In fact, we have $F_1^N v_0 = v_s^{(11)}$, $F_2^N v_0 = v_s^{(44)}$, $F_1^N F_2^N v_0 = v_s^{(33)}$, and $F_1^{2N} F_2^N v_0 = v_s^{(22)}$. On the other hand, $F_{12}^{r_1+r_4} v_0$ and $F_{112}^{r_1+r_4-1} v_0$ belong to $V^{(1)} + V^{(4)}$. So $F_{12}^N v_0, F_{112}^N v_0 \in V^{(1)} + V^{(4)}$ for $r_1 + r_4 < N$.

Therefore, the irreducible module $L(\lambda) = V(\lambda)/(V^{(1)} + V^{(4)})$. But $V^{(1)} + V^{(4)}$ is not the direct sum of vector spaces $V^{(1)}$ and $V^{(4)}$. Furthermore, we have:

Lemma 2.2. Let q be the Nth root of 1 and N is odd. $V(\lambda)$ is the Verma module with the highest weight λ. If the equations (3) have the solutions $0 < r_i < N$ $(1 \leqslant i \leqslant 4)$, such that $r_2 = r_1 + r_4$, $r_3 = r_1 + 2r_4$, then

(i) $V^{(1)} \cap V^{(4)} = V^{(12)} + V^{(43)}$
(ii) $V^{(12)} \cap V^{(43)} = V^{(2)} + V^{(3)}$
(iii) $V^{(2)} \cap V^{(3)} = V^{(1234)}$.

Proof. We have $V^{(1)} \cap V^{(4)} \supset V^{(12)} + V^{(43)}$ by theorem 1.3. But every singular vector contained in $V^{(1)} \cap V^{(4)}$ must be in $V^{(12)} + V^{(43)}$, so (1) holds. Similarly, we can prove (ii) and (iii).

As we know that the character of Verma module $V(\lambda)$ of $U_q(B_2)$ is

$$\mathrm{ch}V(\lambda) = ((1 - t_1)(1 - t_1^2 t_2)(1 - t_1 t_2)(1 - t_2))^{-1}$$

so we have

$$\mathrm{ch}V^{(1)} = t_1^{r_1}\mathrm{ch}V(\lambda) \qquad \mathrm{ch}V^{(4)} = t_2^{r_4}\mathrm{ch}V(\lambda)$$

$$\mathrm{ch}V^{(12)} = t_1^{r_1+2r_4} t_2^{r_4}\mathrm{ch}V(\lambda) \qquad \mathrm{ch}V^{(43)} = t_1^{r_1} t_2^{r_1+r_4}\mathrm{ch}V(\lambda)$$

$$\mathrm{ch}V^{(2)} = t_1^{2(r_1+r_4)} t_2^{r_1+r_4}\mathrm{ch}V(\lambda) \qquad \mathrm{ch}V^{(3)} = t_1^{r_1+2r_4} t_2^{r_1+2r_4}\mathrm{ch}V(\lambda)$$

and

$$\mathrm{ch}V^{(1234)} = t_1^{2(r_1+r_4)} t_2^{r_1+2r_4}\mathrm{ch}V(\lambda).$$

Thus by lemma 2.2 the character of $L(\lambda)$ is

$$\begin{aligned}\mathrm{ch}L(\lambda) &= \mathrm{ch}V(\lambda) - \mathrm{ch}V^{(1)} - \mathrm{ch}V^{(4)} + \mathrm{ch}V^{(12)} + \mathrm{ch}V^{(43)} - \mathrm{ch}V^{(2)} - \mathrm{ch}V^{(3)} + \mathrm{ch}V^{(1234)}\\ &= \mathrm{ch}V(\lambda)(1 - t_1^{r_1} - t_2^{r_4} + t_1^{r_1+2r_4} t_2^{r_4} + t_1^{r_1} t_2^{r_1+r_4} - t_1^{2(r_1+r_4)} t_2^{r_1+r_4}\\ &\quad - t_1^{r_1+2r_4} t_2^{r_1+2r_4} + t_1^{2(r_1+r_4)} t_2^{r_1+2r_4}).\end{aligned}$$

Theorem 2.3. Let q be the Nth root of 1 and N be an odd integer. If the congruence equations (3) have the solutions r_i such that $0 < r_i < N$, $r_2 = r_1 + r_4$ and $r_3 = r_1 + 2r_4$, then the character of the irreducible $U_q(B_2)$-module $L(\lambda)$ is

$$\mathrm{ch}L(\lambda) = \sum_{s=0}^{r_1-1}\sum_{u=1}^{r_4}\sum_{p=0}^{u-1}\sum_{w=0}^{u-1} t_1^s (t_1^2 t_2)^w (t_1 t_2)^{r_4+s-u} t_2^p + \sum_{s=1}^{r_1-1}\sum_{u=0}^{s-1}\sum_{p=0}^{r_4+s-u-1}\sum_{w=0}^{r_4-1} t_1^s (t_1^2 t_2)^w (t_1 t_2)^u t_2^p.$$

Proof. We have

$$1 - t_1^{r_1} - t_2^{r_4} + t_1^{r_1+2r_4} t_2^{r_4} + t_1^{r_1} t_2^{r_1+r_4} - t_1^{2(r_1+r_4)} t_2^{r_1+r_4} - t_1^{r_1+2r_4} t_2^{r_1+2r_4} + t_1^{2(r_1+r_4)} t_2^{r_1+2r_4}$$

$$= (1-t_1)(1-t_1^2 t_2)(1-t_1 t_2)(1-t_2)\left\{\sum_{s=0}^{r_1-1}\sum_{u=1}^{r_4}\sum_{p=0}^{u-1}\sum_{w=0}^{u-1} t_1^s (t_1^2 t_2)^w (t_1 t_2)^{r_4+s-u} t_2^p\right.$$

$$\left. + \sum_{s=1}^{r_1-1}\sum_{u=0}^{s-1}\sum_{p=0}^{r_4+s-u-1}\sum_{w=0}^{r_4-1} t_1^s (t_1^2 t_2)^w (t_1 t_2)^u t_2^p \right\}.$$

Corollary. $\dim L(\lambda) = \frac{1}{6} r_1 r_2 r_3 r_4$.

Proof. By theorem 2.3, we have

$$\dim L(\lambda) = r_1 \sum_{u=1}^{r_4} u^2 + \sum_{s=1}^{r_1-1}\sum_{u=0}^{s-1} r_4(r_4+s-u)$$

$$= \tfrac{1}{6} r_4(r_4+1)(2r_4+1) + r_4^2 \frac{r_1(r_1-1)}{2} + \tfrac{1}{12} r_4(r_1-1) r_1(2r_1-1) + \tfrac{1}{4} r_4 r_1(r_1-1)$$

$$= \tfrac{1}{6} r_1(r_1+r_4)(r_1+2r_4) r_4$$

$$= \tfrac{1}{6} r_1 r_2 r_3 r_4.$$

Acknowledgment

This work is supported by the National Science Foundation of China.

References

[1] De Concini C and Kac V G 1990 Representation of quantum groups of roots 1 *Colloque Dixmier Progress Math. 92 (Birkhaiiser)* pp 471–506

[2] Lusztig G 1990 Quantum groups at root of 1 *Geom. Ded.* **35** 89–114

[3] Dobrev V K 1992 Singular vectors of representations of quantum groups *J. Phys. A: Math. Gen.* **25** 149–60

[4] Zhang R B, Gould M D and Bracken A J 1991 Generalized Gelfand invariants of quantum groups *J. Phys. A: Math. Gen.* **24** 937–43